Subscripts

1 = phase one (either)
2 = phase two (either)
i = interface
o = wall

LITERATURE CITED

1. Dukler, A. E., and Moye Wicks III, in "Modern Chemical Engineering," Andreas Acrivos, ed., Vol. I, Reinhold, New York (1963).
2. Darby, Ron, Ph.D. dissertation, Rice Univ., Houston, Tex. (1961).
3. Hanratty, T. J., and J. M. Engen, *A.I.Ch.E. J.*, **3**, 299 (1957).
4. Shearer, C. J., and R. M. Nedderman, *Chem. Eng. Sci.*, **20**, 671 (1965).
5. Ursell, F., in "Surveys in Mechanics," G. K. Batchelor and R. M. Davies, eds., Cambridge (1956).
6. Charles, M. E., and L. U. Lilleleht, *Can. J. Chem. Eng.*, **44**, 47 (1966).
7. Lockhart, R. W., and R. C. Martinelli, *Chem. Eng. Progr.*, **45**, 39 (1949).
8. Pai, S. I., *J. Appl. Mech.*, **20**, 109 (1953).
9. Brodkey, R. S., *A.I.Ch.E. J.*, **9**, 448 (1963).
10. Stewart, R. W., *J. Fluid Mech.*, **10**, 189 (1961)
11. Keulegan, G. H., *J. Res. Natl. Bureau Standards*, **43**, 487 (1949).

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A Method of Getting Approximate Solutions to the Orr-Sommerfeld Equation for Flow on a Vertical Wall

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A method is presented for getting approximate solutions to the Orr-Sommerfeld equation for free surface flows. The method consists of replacing the velocity, normally a function of distance from the wall, by its value at the free surface while the second derivative of the velocity is kept at its true value. This permits a simple solution to the equation and the eigenvalues can then be determined by a simple and rapid numerical technique. Comparison of this approximate solution for the flow of a thin film on a vertical wall with existing exact numerical solutions and with analytical results valid only for small Reynolds numbers shows the approximation to be quite accurate for most practical values of the parameters and suggests that the method will be useful in investigating the stability of related flows.

A major difficulty in the theory of hydrodynamic stability is the obtaining of accurate solutions to the Orr-Sommerfeld differential equation. In the course of an investigation of finite amplitude wavy flow of a thin film down a vertical wall, we were faced with the problem of finding reasonably accurate solutions to this equation but with a minimum amount of computation. The analytical treatments of Yih (*16, 17*) and of Benjamin (*2*) for infinitesimal disturbances have done much to improve our understanding of the instability of this flow, but their results are limited to flows of low Reynolds number or very low wave number and therefore are not of great practical interest. The numerical techniques of Whitaker (*14, 15*), of Sternling and Barr-David (*10*), and of Sternling and Towell (*11*), while permitting extension of the results to higher Reynolds numbers, are both tedious and time-consuming. In this paper we shall describe a method of getting approximate solutions to the Orr-Sommerfeld equation which seems to be reasonable a priori and which proved to be accurate a posteriori. No attempt is made here to put the approximation on a formal basis or to compute the second term in the approximation; rather, it is our intention to indicate an approximation which is physically reasonable, is easy to carry out, and which proves to be accurate for the present flow. The method should be useful in the investigation of the stability of other flows with a free interface, particularly the flow down a wall of thin films under the influence of surface-active agents.

REVIEW OF PAST WORK

The basic flow whose stability is to be investigated is the parallel flow of a thin film of a Newtonian liquid with constant physical properties under the action of gravity down a flat plate, the film being bounded on the one side by the solid plate and on the other by a free surface subject to capillary forces. By neglecting the air, one finds that the basic velocity profile is a parabola with its vertex at the free surface. Since an excellent review of the literature of film flow, together with a discussion of its practical importance, has been given by Fulford (*4*), we shall review here only the work most pertinent to the present study. Experiments show that this flow is hydrodynamically unstable and that at even small Reynolds numbers the free surface is not smooth but rippled.

Yih (*16*) was the first to attack the problem of the stability of this flow along the lines of classical (linearized) hydrodynamic stability theory. In this treatment (*9*) infinitely small velocity disturbances with a stream function of the form $\phi(y)e^{i\alpha(x-ct)}$ are assumed to be superimposed on the mean parabolic flow. $c = c_R + ic_I$ is the complex wave velocity and is found as an eigenvalue of the problem as a function of the wave number α, the

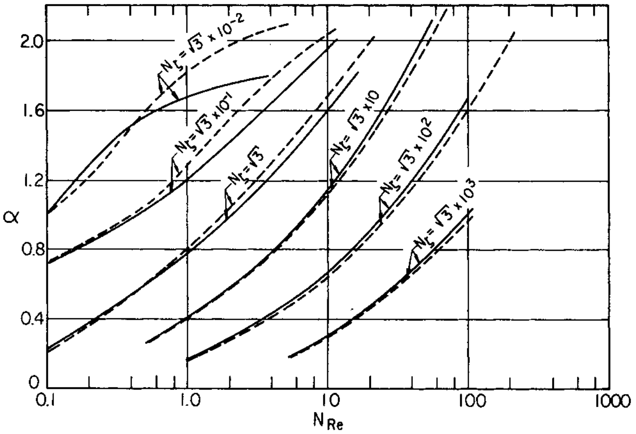

Fig. 1. Neutral stability curves obtained from the modified Orr-Sommerfeld equation (*solid lines*) compared with those calculated by Sterling and Towell from the exact equation (*dashed lines*).

Reynolds number N_{Re}, and a surface tension parameter N_ζ. The sign of the imaginary part of c determines the stability of the flow, for if $c_I < 0$ the flow is hydrodynamically unstable in the sense that infinitesimal disturbance will initially grow exponentially in time; if $c_I > 0$ the disturbance will decay exponentially. The condition $c_I = 0$ imposes a relation among the variables α, N_{Re}, and N_ζ known as the neutral stability curve for which infinitesimal waves are neither damped nor amplified. Yih derived the correct boundary conditions for the Orr-Sommerfeld equation which are reproduced below:

Orr-Sommerfeld equation:

$$\phi'''' - 2\alpha^2\phi'' + \alpha^4\phi = i\alpha N_{Re} \{(\bar{u} - c)(\phi'' - \alpha^2\phi) - \bar{u}''\phi\} \tag{1}$$

Mean velocity profile:

$$\bar{u} = \frac{3}{2}(1 - y^2) \tag{2}$$

Boundary conditions:

$$\text{At } y = 1 \qquad \phi = 0 \tag{3}$$

$$\text{At } y = 1 \qquad \phi' = 0 \tag{4}$$

$$\text{At } y = 0 \quad \left(\frac{3}{2} - c\right)(\phi'' + \alpha^2\phi) + 3\phi = 0 \tag{5}$$

$$\text{At } y = 0 \quad \left(\frac{3}{2} - c\right)(\phi''' - 3\alpha^2\phi') - i\alpha N_{Re}\left(\frac{3}{2} - c\right)^2\phi' - i\alpha^3 N_\zeta N_{Re}{}^{-2/3}\varphi = 0 \tag{6}$$

The Orr-Sommerfeld equation is obtained by substituting the assumed stream function into the Navier-Stokes equations, and by eliminating the pressure by cross differentiation. Boundary conditions (3) and (4) express the no-slip condition at the solid wall. Boundary condition (5) expresses the fact that upon neglecting the influence of the air, there can be no shear stress at the free surface. The last condition balances the normal stress at the free surface with the surface tension and the curvature of the surface.

Benjamin (*2*) sought solutions to the Orr-Sommerfeld equation in the form of power series in y. He took enough terms in this series so that his final results are accurate to the fourth power in α (also regarded as small) and to the second power in αN_{Re}. He found neutral stability curves which began at the αN_{Re} origin, α increasing with increasing N_{Re}. He concluded that a minimum critical Reynolds number does not exist "in the usual sense," but rather that the flow is unstable for all finite Reynolds numbers, a result independent of the presence or absence of surface tension. This situation is to be contrasted with the case of Poiseuille channel flow where the minimum critical Reynolds number is found to be on the order of 6,000 (*9*).

In a more recent article Yih (*17*) sought solutions to the Orr-Sommerfeld equation as a power series in α. Using the first two terms in this series (which are extremely easy to find), Yih was able to confirm the results Benjamin had obtained through his tedious procedure. The principal conflicting features of Yih's and Benjamin's results are those which deal with the physically unrealistic case of zero surface tension.

Bushmanov (*3*) treated the same problem in a manner similar to Benjamin's and found a minimum critical Reynolds number (of about 72 for water), but his boundary conditions at the free surface are not correct, and his calculation is therefore of no value. Bushmanov and Tailby and Portalski (*12*) improved an analysis by Kapitza (*7*) which integrates the equations of motion by assuming that the velocity parallel to the wall is distributed parabolically and that the pressure is a constant across the film. These authors found minimum critical Reynolds numbers. Hanratty and Hershman (*5*), however, using a similar integral approach, found results which agree in form with those of Benjamin.

This problem has also been attacked by using numerical techniques. Thus, in a recent paper by Whitaker (*14*) on the influence of surface-active agents on the stability of the present flow, he solved the Orr-Sommerfeld equation by direct numerical integration with a high-speed computer. The results he obtained for the low Reynolds number flow with finite surface tension in the absence of

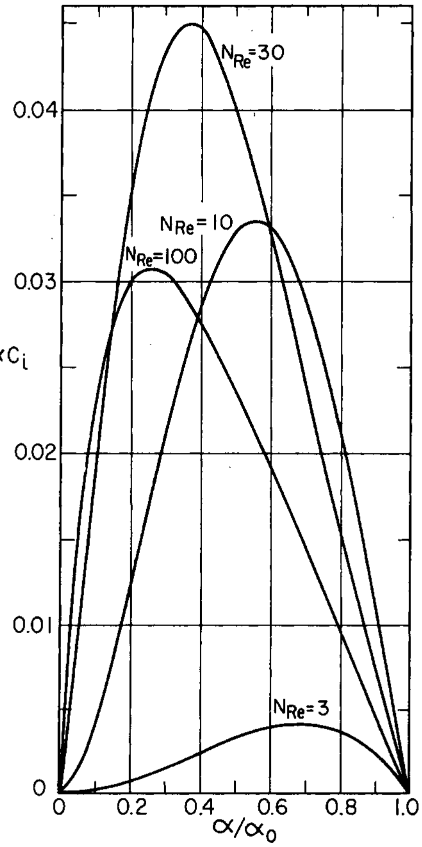

Fig. 2. Representative plots of the amplification factor αc_I vs. the wave number divided by the wave number for neutral stability for various Reynolds numbers of water at 20°C. N_ζ = 4,887.

surfactants were in complete agreement with those of Benjamin (2) and of Yih (17). Whitaker has extended these calculations elsewhere (15). Sternling and Barr-David (10) have programmed Benjamin's power series technique; terms were added to the series until the final term did not affect the first eight significant figures in ϕ'''. A transcendental equation is then solved for the eigenvalues. Again good agreement for the low Reynolds number analytical results was found.

ANALYSIS AND PREDICTIONS

Yih (17) has pointed out that the stability characteristics of the falling film are governed by the surface or kinematic waves. Indeed, treating this flow as an example of the type of kinematic waves discussed by Lighthill and Whitham (8) yields the correct value of the wave velocity in the limit of very long wavelength. The fact that c occurs in the surface boundary conditions is what enables Yih to make his expansion in small α. If it is true that surface waves are governing, then dropping the y^2 from $\bar{u}$ in Equation (1) should not alter the eigenvalues greatly, because near the surface $\bar{u} = 3/2$ is a very good approximation. We therefore consider Equation (1) with $\bar{u}'' = -3$ and $\bar{u} = 3/2$, which we call the modified Orr-Sommerfeld equation:

$$\phi'''' - 2\alpha^2\phi'' + \alpha^4\phi = i\alpha N_{Re}\left\{\left(\frac{3}{2} - c\right)(\phi'' - \alpha^2\phi) + 3\phi\right\} \quad (7)$$

Boundary conditions (3) and (6) remain unchanged.

A quick check of this equation in a Yih type of expression in small α shows that to the lowest order c_R is exactly the same as the from the full equation and that α is about 2½% off along the neutral stability curve. The real advantage in Equation (7) is that the coefficients are constant so that the solution to the differential equation can be written immediated in terms of tabulated functions. Thus

$$\phi = A \sin \beta_1 y + B \cos \beta_1 y + C \sin \beta_2 y + D \cos \beta_2 y \quad (8)$$

where

$$\beta_1 = \left\{-\alpha^2 - ½ i\alpha N_{Re}\left(\frac{3}{2} - c\right) + \left\{3i\alpha N_{Re} + \left[½\ i\alpha N_{Re}\left(\frac{3}{2} - c\right)\right]^2\right\}^{1/2}\right\}^{1/2}$$

and

$$\beta_2 = \left\{-\alpha^2 - ½ i\alpha N_{Re}\left(\frac{3}{2} - c\right) - \left\{3i\alpha N_{Re} + \left[½ i\alpha N_{Re}\left(\frac{3}{2} - c\right)\right]^2\right\}^{1/2}\right\}^{1/2} \quad (9)$$

When this solution is substituted into the four linear homogeneous boundary conditions, a nontrivial solution for the constants A, B, C, D exists only if the determinant of the coefficient vanishes. This leads to the following secular equation:

$$F(\alpha, N_{Re}, N_\zeta, c) = \begin{vmatrix} \sin \beta_1 & \cos \beta_1 & \sin \beta_2 & \cos \beta_2 \\ \beta_1 \cos \beta_1 & -\beta_1 \sin \beta_1 & \beta_2 \cos \beta_2 & -\beta_2 \sin \beta_2 \\ 0 & \left(\frac{3}{2} - c\right)(\alpha^2 - \beta_1^2) + 3 & 0 & \left(\frac{3}{2} - c\right)(\alpha^2 - \beta_2^2) + 3 \\ \left(\frac{3}{2} - c\right)\beta_1\left\{i\alpha N_{Re}\left(\frac{3}{2} - c\right) - 3\alpha^2 + \beta_1^2\right\} & i\alpha^3 N_\zeta N_{Re}^{-2/3} & \left(\frac{3}{2} - c\right)\beta_2\left\{i\alpha N_{Re}\left(\frac{3}{2} - c\right) - 3\alpha^2 + \beta_2^2\right\} & i\alpha^3 N_\zeta N_{Re}^{-2/3} \end{vmatrix} = 0 \quad (10)$$

F, of course, is still a very complicated function of the flow parameters, but it is an *explicit* function of α, N_{Re}, N_ζ, and c. A computer program was written to calculate c, given α, N_{Re}, and N_ζ via the Newton-Raphran numerical iteration. The program, together with an initial guess for c, is given in the thesis by Anshus (1). To illustrate the ease of calculating eigenvalues, c was calculated to five significant figures in real and imaginary parts for one hundred evenly spaced values of α between $\alpha = 0$ and the neutral stability curve for thirty values of the Reynolds number between 1 and 1,000 by using the value of N_ζ corresponding to pure water at 20°C., a total of 3,000 calculations. This calculation took about 6 min. on an IBM 7094.

As a check on the accuracy of this method, neutral stability curves have been calculated for a number of surface tension parameters and have been compared with neutral stability curves calculated by a numerical solution of the exact Orr-Sommerfeld equation as supplied by Sternling and Towell (11). The results are shown in Figure 1. The two calculations show good agreement for surface tension parameters which correspond to fluids of greatest practical interest. N_ζ is about 5,000 for water and about 5 for mineral oils with a viscosity of 1 poise. In Figure 2 we show the amplification factor αc_I as a function of the wave number (relative to the wave number for neutral stability) for selected values of the Reynolds numbers. The curves are for water at 20°C., that is, $N_\zeta = 4{,}887$. It is to be noted that for Reynolds numbers above 10 the curves are moderately well peaked, so that one should expect fairly well-defined wavelengths and wave velocities corresponding to the most rapidly growing wave. Thus in Figure 3 are plotted the wavelengths measured by Tailby and Portalski (12, 13) on a water film at various Reynolds numbers. These are compared with the wavelength of the most highly amplified wave. Also plot-

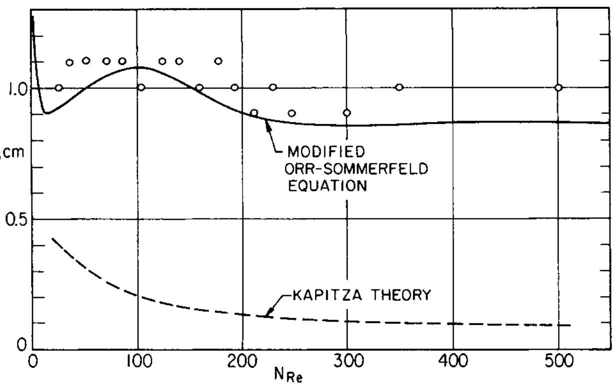

Fig. 3. Wavelength of the most rapidly amplified wave vs. the Reynolds number as calculated from the modified Orr-Sommerfeld equation (*solid curve*) compared with the wavelength observed experimentally by Tailby and Portalski for water and with the wave length as predicted by the Portalski-modified Kapitza theory (*dashed curve*).

ted is the wavelength predicted by Tailby and Portalski from a slightly modified Kapitza analysis. The two theories give quite different predictions for Reynolds numbers greater than about 10. Although the data are too scattered to check the detailed shape of the curve, the present calculation at least predicts the observed magnitude of λ throughout the entire Reynolds number range investigated. In Figure 4 the dimensionless wave velocity for the most highly amplified wave for water at 20°C. is plotted vs. the Reynolds number. Also shown are the experimental wave velocities of Jones and Whitaker (*6*) and the wave velocity predicted by Whitaker (*15*), who used a numerical direct integration of the exact Orr-Sommerfeld equation. Again, the agreement between both the experiments and Whitaker's much more complicated calculation and the present calculations is thought to be quite good.

As can be seen from Figure 2 the maximum amplification rate αc_I has a maximum when regarded as a function of the Reynolds number. This means that the rate of growth of small disturbance increases with increasing Reynolds number for low Reynolds numbers (less than ~25 for water) but decreases with increasing Reynolds number at high Reynolds numbers. At present we do not have a physical interpretation of this prediction or the data to check it. Figure 5 gives a plot of the maximum amplification rate as a function of the Reynolds number, again for the case of water at 20°C. This plot is included here because it was found to be useful in discussing the

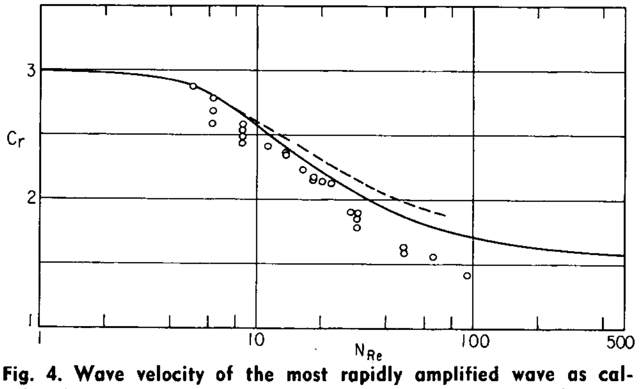

Fig. 4. Wave velocity of the most rapidly amplified wave as calculated from the modified Orr-Sommerfeld equation (*solid curve*) compared with wave velocity data measured by Jones and Whitaker for water and with the wave velocity calculated by Whitaker with a numerical integration of the Orr-Sommerfeld equation (*dashed curve*).

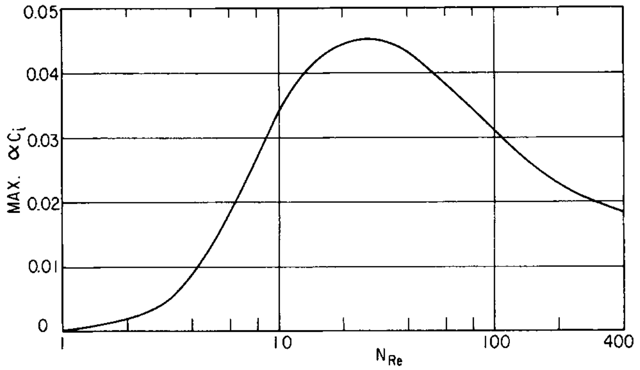

Fig. 5. Maximum amplification factor αc_I vs. Reynolds number for water at 20°C. as calculated from the modified Orr-Sommerfeld equation.

increase of mass transfer to the rippled flow, considerations which will be presented for publication shortly.

In his second paper Yih poses the interesting question of where are the vestiges of shear waves which are responsible for the instability of Poiseuille channel flow? One important characteristic of the shear waves is the occurrence of a point within the flow where the local velocity is equal to the wave velocity. The calculations here indicate that for Reynolds number up to 1,000 for water the wave velocity is always greater than the surface velocity, that is, the maximum fluid velocity. Apparently this is why the stability is dominated by surface waves and, we think, the reason why the present approximation proves to be so accurate. As the Reynolds number increases above 1,000 the wave velocity appears to approach the surface velocity; whether the two become equal for finite Reynolds number or only for $N_{Re} = \infty$ is a matter we cannot answer yet. For flows with small N_ζ the wave velocity along the neutral stability curve approaches the surface velocity at much lower Reynolds numbers than for the case of water. The approximation appears to break down when c_R approaches 3/2, that is, the surface velocity, and this may explain the discrepancy shown in Figure 1 between the present predictions and those of Sternling and Towell for the neutral stability curves for flows with small N_ζ.

In conclusion, the approximate solution to the Orr-Sommerfeld equation applied here to the flow of a thin film on a vertical wall proves to be quite accurate for most practical values of the parameters as judged by comparing its results with those of much longer and more complex numerical methods and with analytical results valid only for small Reynolds numbers. The approximation should be useful in the investigation of the stability of other flows with a free surface. In view of its success the approximation warrants further study aimed at putting it on a formal basis.

ACKNOWLEDGMENT

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NOTATION

$c = c_R + ic_I$ = dimensionless complex wave velocity, c^*/u_m
g = acceleration of gravity, cm./sec.^{2}
h = undisturbed film thickness, cm.
i = imaginary unit
N_{Re} = Reynolds number, hu_m/ν
N_ζ = dimensionless surface tension parameter, $\sigma 3^{1/3}/\rho g^{1/3}\nu^{4/3}$
t = dimensionless time, $t^* u_m/h$

$\bar{u} = 3/2\ [1 - y^2]$ = dimensionless mean velocity profile, $\bar{u}^*/u_m$

u_m = average velocity in the film, cm./sec.

x = dimensionless Cartesian coordinate in direction of flow, x^*/h

y = dimensionless Cartesian coordinate normal to the plate, y^*/h

α = dimensionless wave number, $2\pi h/\lambda$

λ = wavelength, cm.

ν = kinematic viscosity, sq.cm./sec.

ρ = density, g./cc.

σ = surface tension, dyne/cm.

$\phi(y)$ = dimensionless amplitude of disturbance stream function

LITERATURE CITED

1. Anshus, B. E., Ph.D. thesis, Univ. California, Berkeley (1965).
2. Benjamin, T. B., *J. Fluid Mech.*, **2**, 554 (1957).
3. Bushmanov, C. K., *Soviet Phys. JETP*, **12**, 873 (1961).
4. Fulford, G. D., "Advances in Chemical Engineering," Vol. 5, Academic Press, New York (1964).
5. Hanratty, T. J., and A. Hershman, *A.I.Ch.E. J.*, **7**, 488 (1961).
6. Jones, L. O., and Stephen Whitaker, *ibid.*, **12**, No. 3, 525-529 (1966).
7. Kapitza, P. L., *J. Expt. Theor. Phys. USSR*, **18**, 3 (1948).
8. Lighthill, M. J., and G. B. Whitham, *Proc. Roy. Soc. (London)*, **A229**, 281 (1955).
9. Lin, C. C., "The Theory of Hydrodynamic Stability," Cambridge Univ. Press, London (1955).
10. Sternling, C. V., and F. H. Barr-David, unpublished.
11. Sternling, C. V., and G. D. Towell, private communication.
12. Tailby, S. R., and S. Portalski, *Trans. Inst. Chem. Engrs.*, **38**, 324 (1960).
13. *Ibid.*, **40**, 114 (1962).
14. Whitaker, Stephen, *Ind. Eng. Chem. Fundamentals*, **3**, 132 (1964).
15. ———, unpublished.
16. Yih, C. S., "Proc. 2 U. S. Natl. Congr. Applied Mech.," p. 623, Am. Soc. Mech. Engrs., New York (1954).
17. ———, *Phys. Fluids*, **6**, 321 (1963).

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Liquid Surface Area Effects in a Wetted-Wall Column

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Vaporization of methanol and carbon tetrachloride into air and carbon tetrachloride into helium was carried out in a counterflow wetted-wall column at 10° to 20°F. These data show an influence of liquid rate upon the mass transfer coefficients. Liquid rate influence has been empirically accounted for by a function of surface area increase caused by rippling of the liquid film.

Kafesjian et al. (*5, 6*) showed that the liquid flow rate must be considered as a variable in the correlation of gas phase mass transfer in wetted-wall columns. Except for very low liquid Reynolds numbers, thin films do not flow smoothly down a vertical wall. The surface becomes covered with ripples, increasing the interfacial contact area; in fact four different flow regimes have been defined (*2, 4, 11, 19*). Most of the data concerning wetted-wall correlations and performance whch were published prior to 1960 have been reviewed previously (*5, 6*).

More recently, Parrish (*10*) obtained data by vaporizing methanol into nitrogen and also into helium. Sebulsky (*14, 18*) vaporized benzene into nitrogen and also the binary benzene-acetone into helium. Strumillo and Porter (*15*) showed that the liquid rate increased the rate of mass transfer for carbon tetrachloride vaporizing in a wetted-wall column. Pertinent studies of thin liquid films in rippling flow have been published by Portalski et al. (*11, 12, 16, 17*). An excellent review of phenomena associated with flow of liquids in thin films has been given by Fulford (*3a*).

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EXPERIMENTAL APPARATUS AND PROCEDURE

In the present investigation data have been obtained on low-temperature (10° to 20°F.) vaporization of methanol and carbon tetrachloride into air and carbon tetrachloride into helium in a counterflow wetted-wall column. This apparatus was similar to that described by Kafesjian (*6*). Complete details can be found in reference *13*. Temperatures of the inlet and exit gas and liquid streams were measured by fine wire copper-constantan thermocouples to an accuracy of ± 0.05°F. Exit compositions of the gas stream were measured by a thermal conductivity cell for the carbon tetrachloride runs. Wet and dry bulb thermometry was used for the methanol.

CALCULATION PROCEDURE

The mass transfer coefficient may be properly defined for small rates of mass transfer in a differential section by the equation

$$\frac{G}{M_{\text{avg}}\,P}\,dp = k_G a(p^* - p)dZ \qquad (1)$$

or

$$\int_1^2 \frac{M_{\text{avg}}P\,k_G a dZ}{G} = \int_1^2 \frac{dp}{p^* - p} \qquad (1a)$$